

ON UPPER BOUNDS OF ARITHMETIC DEGREES

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ABSTRACT. Let X be a smooth projective variety over $\overline{\mathbb{Q}}$, and $f : X \dashrightarrow X$ be a dominant rational map. Let δ_f be the first dynamical degree of f and $h_X : X(\overline{\mathbb{Q}}) \rightarrow [1, \infty)$ be a Weil height function on X associated with an ample divisor on X . We prove several inequalities which give upper bounds of the sequence $(h_X(f^n(P)))_{n \geq 0}$ where P is a point of $X(\overline{\mathbb{Q}})$ whose forward orbit by f is well-defined. As a corollary, we prove that the upper arithmetic degree is less than or equal to the first dynamical degree; $\overline{\alpha}_f(P) \leq \delta_f$. Furthermore, if the Picard number of X is one, f is algebraically stable and $\delta_f > 1$, we prove that the limit defining canonical height $\lim_{n \rightarrow \infty} h_X(f^n(P))/\delta_f^n$ converges.

1. INTRODUCTION

Let X be a smooth projective variety over an algebraic closure $\overline{\mathbb{Q}}$ of the field of rational numbers \mathbb{Q} and $f : X \dashrightarrow X$ a dominant rational map defined over $\overline{\mathbb{Q}}$. The (first) dynamical degree δ_f of f is a measure of the geometric complexity of the iterates f^n of f . The dynamical degree of a dominant rational self-map on an arbitrary smooth projective variety over \mathbb{C} is defined by Dinh-Sibony in [3, 4] using Kähler form on X . The alternating definition is introduced by Diller-Favre in [2] using the linear map f^* induced on the Neron-Severi group of X . The first dynamical degree is a birational invariants of f and is an important tool for the study of dynamics of algebraic varieties.

On the other hand, by studying the asymptotic behavior on n of the height of iterations $f^n(P)$ where $P \in X(\overline{\mathbb{Q}})$ is a point whose f -orbit is well-defined, Silverman introduced in [14] the arithmetic degree of the orbit. It measures the arithmetic complexity of f -orbits. In [14], he expects the coincidence of the dynamical degree and the arithmetic degree of a Zariski dense orbit. A refined version of this conjecture was formulated by Kawaguchi and Silverman in [9]. Related topics are studied in [8, 9, 10, 13, 14, 15].

In this paper, we give upper bounds of heights of $f^n(P)$ in terms of δ_f or the spectral radius of the linear map f^* induced on the Neron-Severi group of X . The main theorem of this paper is Theorem 1.4 below. Actually, this theorem is stated as Theorem 1 in [9]. However, K. Sano pointed out that the proof of Theorem 1 in [9] was not correct (cf. Remark 1.5). In this paper, we give a correct proof of Theorem 1 in [9].

To give a precise statements of our main results, we recall the definition of the dynamical and arithmetic degree.

The first dynamical degree

Let $N^1(X)$ be the group of divisors on X modulo numerical equivalence. Since X is smooth, this is equal to the group of codimension one cycles modulo numerical equivalence. The group $N^1(X)$ is a free \mathbb{Z} -module of finite rank. We write $N^1(X)_{\mathbb{R}}$

for $N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$. The map f induces the pull-back homomorphism $f^* : N^1(X) \rightarrow N^1(X)$.

Definition 1.1.

- (1) For an endomorphism φ of a finite dimensional real vector space, the maximum of the absolute values of eigenvalues of φ is called the spectral radius of φ and denoted by $\rho(\varphi)$.
- (2) The first dynamical degree δ_f of f is defined as follows.

$$\delta_f = \lim_{n \rightarrow \infty} \rho((f^n)^* : N^1(X)_{\mathbb{R}} \rightarrow N^1(X)_{\mathbb{R}})^{1/n}.$$

It is known that the limit defining δ_f exists and this is a birational invariant. Note that $\delta_f \geq 1$ since f is dominant and $(f^n)^*$ is a homomorphism of the \mathbb{Z} -lattice $N^1(X)$.

The arithmetic degree

For a projective space \mathbb{P}^N with given coordinates, a function on $\mathbb{P}^N(\overline{\mathbb{Q}})$ called the absolute logarithmic height function is defined (see [1, 6, 12] for the definition). The height function does not depend on the choice of coordinates up to bounded functions. If we fix an embedding $X \rightarrow \mathbb{P}^N$, we have a height function h_X on $X(\overline{\mathbb{Q}})$.

We write $h_X^+ = \max\{h_X, 1\}$. Let I_f be the indeterminacy locus of f . We want to consider the orbit of a point by f , so we set

$$X_f(\overline{\mathbb{Q}}) = \{P \in X(\overline{\mathbb{Q}}) \mid f^n(P) \notin I_f \text{ for all } n \geq 0\}.$$

Definition 1.2. Let $P \in X_f(\overline{\mathbb{Q}})$. The arithmetic degree of P is

$$\alpha_f(P) = \lim_{n \rightarrow \infty} h_X^+(f^n(P))^{1/n}$$

if the limit exists. Since it is not known whether the limit always exists, the following invariants are introduced by S. Kawaguchi and J. H. Silverman in [9].

$$\overline{\alpha}_f(P) = \limsup_{n \rightarrow \infty} h_X^+(f^n(P))^{1/n}$$

$$\underline{\alpha}_f(P) = \liminf_{n \rightarrow \infty} h_X^+(f^n(P))^{1/n}.$$

These are called the upper and lower arithmetic degree of P . By definition, $1 \leq \underline{\alpha}_f(P) \leq \overline{\alpha}_f(P)$.

In [9], Kawaguchi and Silverman proposed the following very deep conjecture.

Conjecture 1.3. Let $P \in X_f(\overline{\mathbb{Q}})$.

- (1) The limit defining $\alpha_f(P)$ exists.
- (2) The arithmetic degree $\alpha_f(P)$ is an algebraic integer.
- (3) The collections of arithmetic degrees $\{\alpha_f(Q) \mid Q \in X_f(\overline{\mathbb{Q}})\}$ is a finite set.
- (4) If the forward orbit $\mathcal{O}_f(P) = \{f^n(P) \mid n = 0, 1, 2, \dots\}$ is Zariski dense in X , then $\alpha_f(P) = \delta_f$.

For example, this conjecture is proved in the following situations:

- (1) $N^1(X)_{\mathbb{R}} = \mathbb{R}$ and f is a morphism [8].
- (2) $f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ is a monomial map and $P \in \mathbb{G}_m^N(\overline{\mathbb{Q}})$ [14].
- (3) X is a surface and f is an automorphism [7].
- (4) $X = \mathbb{P}^N$ and f is a rational map extending a regular affine automorphism [8].

- (5) X is an abelian variety [10, 15].

When f is a morphism, the first three parts of this conjecture are proved by Kawaguchi and Silverman in [10] (cf. Remark 1.7). Sano studies in [13] about this conjecture for endomorphisms on products of varieties such that the conjecture holds for every factor. See [8, 13, 14] for more details about this conjecture.

The main theorem of this paper is the following.

Theorem 1.4. *Let $f : X \dashrightarrow X$ be a dominant rational map defined over $\overline{\mathbb{Q}}$. For any $\epsilon > 0$, there exists $C > 0$ such that*

$$h_X^+(f^n(P)) \leq C(\delta_f + \epsilon)^n h_X^+(P)$$

for all $n \geq 0$ and $P \in X_f(\overline{\mathbb{Q}})$. In particular, for any $P \in X_f(\overline{\mathbb{Q}})$, we have

$$\overline{\alpha}_f(P) \leq \delta_f.$$

Remark 1.5. This theorem is stated as Theorem 1 in [9]. However, their proof is incorrect. Precisely, in the proof of Theorem 24 (Theorem 1) in [9], the constant C_1 and therefore C_8 depends on m . Thus one can not conclude the equality $\lim_{m \rightarrow \infty} (C_8 r m^r)^{1/ml} = 1$ which is a key in the argument of the proof in [9].

If f is a morphism, we have the following slightly stronger inequalities.

Theorem 1.6. *Let $f : X \rightarrow X$ be a surjective morphism. Let $r = \dim N^1(X)_{\mathbb{R}}$ be the Picard number of X .*

- (1) *When $\delta_f = 1$, there exists a constant $C > 0$ such that*

$$h_X^+(f^n(P)) \leq C n^{2r+2} h_X^+(P)$$

for all $n \geq 1$ and $P \in X(\overline{\mathbb{Q}})$.

- (2) *Assume that $\delta_f > 1$. Then there exists a constant $C > 0$ such that*

$$h_X^+(f^n(P)) \leq C n^r \delta_f^n h_X^+(P)$$

for all $n \geq 1$ and $P \in X(\overline{\mathbb{Q}})$.

Remark 1.7. In [10], Kawaguchi and Silverman prove a similar inequality under the same assumption of Theorem 1.6. Moreover, they prove that the arithmetic degree $\alpha_f(P)$ exists and is equal to one of the eigenvalues of the linear map $f^* : N^1(X)_{\mathbb{R}} \rightarrow N^1(X)_{\mathbb{R}}$. Thus for a surjective morphism f , the first three parts of Conjecture 1.3 and the inequality $\alpha_f(P) \leq \delta_f$ are proved.

If the Picard rank of X is one, we have the following stronger inequalities.

Theorem 1.8. *Let X be a smooth projective variety with the Picard number one. Let $f : X \dashrightarrow X$ be a dominant rational map.*

- (1) *For a positive integer $k > 0$, there exists a constant $C > 0$ such that*

$$h_X^+(f^n(P)) \leq C n^2 \rho((f^k)^*)^{n/k} h_X^+(P)$$

for all $P \in X_f(\overline{\mathbb{Q}})$ and $n \geq 1$.

- (2) *Let $k > 0$ be a positive integer. Assume that $\rho((f^k)^*) > 1$. Then there exists a constant $C > 0$ such that*

$$h_X^+(f^n(P)) \leq C \rho((f^k)^*)^{n/k} h_X^+(P)$$

for all $P \in X_f(\overline{\mathbb{Q}})$ and $n \geq 0$.

A dominant rational map f is said to be algebraically stable if $(f^n)^* = (f^*)^n : N^1(X)_{\mathbb{R}} \rightarrow N^1(X)_{\mathbb{R}}$ for all $n > 0$. In this case, $\delta_f = \rho(f^*)$. As an application of the computation in the proof of Theorem 1.4, we can show the following.

Proposition 1.9. *Assume that the Picard rank of X is one and let $f : X \dashrightarrow X$ be an algebraically stable dominant rational map with $\delta_f > 1$. Then the limit*

$$\hat{h}_{X,f}(P) = \lim_{n \rightarrow \infty} \frac{h_X(f^n(P))}{\delta_f^n}$$

exists for all $P \in X_f(\overline{\mathbb{Q}})$.

Proposition 1.10. *Let X be a smooth projective variety over $\overline{\mathbb{Q}}$. Let $f : X \dashrightarrow X$ be a dominant rational self-map defined over $\overline{\mathbb{Q}}$. Assume $\delta_f > 1$ and there exists a nef \mathbb{R} -divisor H on X such that $f^*H \equiv \delta_f H$. Fix a height function h_H associated with H . Then for any $P \in X_f(\overline{\mathbb{Q}})$, if the sequence $(h_H(f^n(P))/\delta_f^n)_{n \geq 0}$ is bounded below, the limit*

$$\hat{h}_{X,f}(P) = \lim_{n \rightarrow \infty} \frac{h_H(f^n(P))}{\delta_f^n}$$

exists.

Remark 1.11. The condition “ $(h_H(f^n(P))/\delta_f^n)_{n \geq 0}$ is bounded below” might be automatically satisfied, but currently this hypothesis can not be removed.

The function $\hat{h}_{X,f}$ is the function which is called the canonical height function in [14].

We prove Theorem 1.6 in §2, Theorem 1.4 in §3, Theorem 1.8 and Proposition 1.9, 1.10 in §4. In the proof of Theorem 1.8 and Proposition 1.9, we use the computation in the proof of Theorem 3.2 in §3. Although the proof of Theorem 1.6 is not used in later sections for the most part, it is helpful to understand the computation in §3.

In this paper, we give a method to estimate $h_H(f^n(P))$ in terms of behavior of f on the group $N^1(X)_{\mathbb{R}}$ of numerically equivalence classes of divisors by controlling error terms arising from divisors numerically equivalent to zero. We give an expression of error terms as a linear combinations of fixed height functions whose coefficients can be controlled easily.

Remark 1.12. Let D be an \mathbb{R} -divisor on X . Then D determines a unique (logarithmic) Weil height function h_D up to bounded functions as follows. When D is very ample integral divisor, h_D is the composite of the embedding by $|D|$ and the height on the projective space. For general D , we write

$$(1) \quad D = \sum_{i=1}^m a_i H_i$$

where a_i are real numbers and H_i are very ample divisors. Then we define

$$h_D = \sum_{i=1}^m a_i h_{H_i}.$$

The function h_D does not depend on the choice of the representation (1) up to bounded function (see [1, 6, 12] for the detail). We call any representative of the class $h_D \bmod (\text{bounded functions})$ a height function associated with D . We call a height function associated with an ample divisor an ample height function.

In the above definition, theorems and proposition, we fix a height function h_X . Actually, for the definition of arithmetic degree, we can replace h_X by any ample height functions. Also, the above theorems and proposition are valid for all ample height functions h_X . Indeed, note that for any ample height functions h, h' , there exists a positive number c such that

$$ch^+ \geq h'^+, \quad ch'^+ \geq h^+$$

on $X(\overline{\mathbb{Q}})$. Thus, for the proof of the above theorems, it is enough to prove them for a particular ample height function.

Remark 1.13. All of the results and arguments in this paper remain valid without change for other ground fields of characteristic 0 with a set of non-trivial absolute values satisfying the product formula. For positive characteristic, see Appendix B.

2. ENDOMORPHISM CASE

We first treat the case where f is a morphism. The purpose of this section is to prove the following theorem.

Theorem 2.1 (Theorem 1.6). *Let X be a smooth projective variety over $\overline{\mathbb{Q}}$ and $f : X \rightarrow X$ be a surjective morphism defined over $\overline{\mathbb{Q}}$ with first dynamical degree δ_f . Let $r = \dim N^1(X)_{\mathbb{R}}$ be the Picard number of X . Fix an ample height function h_X on X .*

- (1) *When $\delta_f = 1$, there exists a constant $C > 0$ such that*

$$h_X^+(f^n(P)) \leq Cn^{2r+2}h_X^+(P)$$

for all $n \geq 1$ and $P \in X(\overline{\mathbb{Q}})$.

- (2) *Assume that $\delta_f > 1$. Then there exists a constant $C > 0$ such that*

$$h_X^+(f^n(P)) \leq Cn^r\delta_f^n h_X^+(P)$$

for all $n \geq 1$ and $P \in X(\overline{\mathbb{Q}})$.

Proof. Let D_1, \dots, D_r be \mathbb{R} -divisors which form a basis for $N^1(X)_{\mathbb{R}}$. Let H be an ample divisor on X such that $H + D_i, H - D_i$ ($i = 1, \dots, r$) are ample. For \mathbb{R} -divisors α, β , $\alpha \equiv \beta$ means α and β are numerically equivalent. Let $f^*D_i \equiv \sum_{k=1}^r a_{ki}D_k$, and $A = (a_{ki})_{k,i}$. We can write $H \equiv \sum_{i=1}^r c_i D_i$. Then

$$f^*H \equiv \sum_{j=1}^r \sum_{k=1}^r c_j a_{kj} D_k = \left\langle A \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{pmatrix}, \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_r \end{pmatrix} \right\rangle = \langle A\vec{c}, \vec{D} \rangle.$$

Let

$$(2) \quad E = f^*H - \langle A\vec{c}, \vec{D} \rangle$$

$$(3) \quad E_i = f^*D_i - \sum_{k=1}^r a_{ki}D_k.$$

Then

$$\vec{E} = \begin{pmatrix} E_1 \\ E_2 \\ \vdots \\ E_r \end{pmatrix} = f^* \vec{D} - {}^t A \vec{D}.$$

Note that E, E_i are numerically zero.

The choice of Height functions.

First, we take and fix height functions h_{D_1}, \dots, h_{D_r} associated with D_1, \dots, D_r . Next, we take and fix a height function h_H associated with H so that $h_H \geq 1$, $h_H \geq |h_{D_i}|$ ($i = 1, \dots, r$). Then $h_{D_i} \circ f$, $h_H \circ f$ are height functions associated with $f^* D_i$ and $f^* H$. We write

$$\mathbf{h}_{\vec{D}} = \begin{pmatrix} h_{D_1} \\ h_{D_2} \\ \vdots \\ h_{D_r} \end{pmatrix}.$$

We define

$$(4) \quad h_E = h_H \circ f - \langle A \vec{c}, \mathbf{h}_{\vec{D}} \rangle$$

$$(5) \quad \mathbf{h}_{\vec{E}} = \begin{pmatrix} h_{E_1} \\ h_{E_2} \\ \vdots \\ h_{E_r} \end{pmatrix} = \mathbf{h}_{\vec{D}} \circ f - {}^t A \mathbf{h}_{\vec{D}}.$$

Then, by (2)(3), h_E and h_{E_i} are height functions associated with E and E_i . Now, since E, E_i are numerically zero, there exists a constant $C > 0$ such that for all $Q \in X(\overline{\mathbb{Q}})$

$$(6) \quad |h_E(Q)| \leq C \sqrt{h_H(Q)}$$

$$(7) \quad |h_{E_i}(Q)| \leq C \sqrt{h_H(Q)} \quad i = 1, \dots, r.$$

See for example ([6] Theorem B.5.9).

Let us begin the estimation of $h_H(f^n(P))$. Let $P \in X(\overline{\mathbb{Q}})$ be an arbitrary point. Then for $n \geq 1$

$$\begin{aligned}
h_H(f^n(P)) &= (h_H \circ f)(f^{n-1}(P)) - \langle A\vec{c}, \mathbf{h}_{\bar{D}} \rangle(f^{n-1}(P)) \\
&\quad + \langle A\vec{c}, \mathbf{h}_{\bar{D}} \circ f \rangle(f^{n-2}(P)) - \langle A^2\vec{c}, \mathbf{h}_{\bar{D}} \rangle(f^{n-2}(P)) \\
&\quad + \cdots \\
&\quad + \langle A^{n-2}\vec{c}, \mathbf{h}_{\bar{D}} \circ f \rangle(f(P)) - \langle A^{n-1}\vec{c}, \mathbf{h}_{\bar{D}} \rangle(f(P)) \\
&\quad + \langle A^{n-1}\vec{c}, \mathbf{h}_{\bar{D}} \circ f \rangle(P) \\
&= h_E(f^{n-1}(P)) \\
&\quad + \langle A\vec{c}, {}^t A \mathbf{h}_{\bar{D}} + \mathbf{h}_{\bar{E}} \rangle(f^{n-2}(P)) - \langle A^2\vec{c}, \mathbf{h}_{\bar{D}} \rangle(f^{n-2}(P)) \\
&\quad + \cdots \\
&\quad + \langle A^{n-2}\vec{c}, {}^t A \mathbf{h}_{\bar{D}} + \mathbf{h}_{\bar{E}} \rangle(f(P)) - \langle A^{n-1}\vec{c}, \mathbf{h}_{\bar{D}} \rangle(f(P)) \\
&\quad + \langle A^{n-1}\vec{c}, {}^t A \mathbf{h}_{\bar{D}} + \mathbf{h}_{\bar{E}} \rangle(P) \quad \text{by (4)(5)} \\
&= h_E(f^{n-1}(P)) \\
&\quad + \langle A\vec{c}, \mathbf{h}_{\bar{E}} \rangle(f^{n-2}(P)) \\
&\quad + \cdots \\
&\quad + \langle A^{n-2}\vec{c}, \mathbf{h}_{\bar{E}} \rangle(f(P)) \\
&\quad + \langle A^{n-1}\vec{c}, \mathbf{h}_{\bar{E}} \rangle(P) + \langle A^n\vec{c}, \mathbf{h}_{\bar{D}} \rangle(P).
\end{aligned}$$

For a real $r \times r$ -matrix $M = (m_{ij})$, we put $\|M\| = \max_{1 \leq i \leq r} \{|c_i|\} r^2 \max_{i,j} \{|m_{ij}|\}$. This is a norm for the matrix space. By (6)(7)

$$|\langle A^m\vec{c}, \mathbf{h}_{\bar{E}} \rangle(Q)| \leq \|A^m\| C \sqrt{h_H(Q)} \quad \text{for } Q \in X(\overline{\mathbb{Q}}).$$

Thus

$$\begin{aligned}
(8) \quad h_H(f^n(P)) &\leq C \left(\sqrt{h_H(f^{n-1}(P))} + \|A\| \sqrt{h_H(f^{n-2}(P))} + \cdots \right. \\
&\quad \left. + \|A^{n-2}\| \sqrt{h_H(f(P))} + \|A^{n-1}\| \sqrt{h_H(P)} \right) + \|A^n\| h_H(P).
\end{aligned}$$

For simplicity, we write $\delta = \delta_f$. Let $\rho(f^*)$ be the spectral radius of the linear map $f^* : N^1(X)_{\mathbb{R}} \rightarrow N^1(X)_{\mathbb{R}}$. Let $\rho(A)$ be the spectral radius of the matrix A . Since f is a morphism, we have $\delta = \rho(f^*) = \rho(A) = \lim_{n \rightarrow \infty} \|A^n\|^{1/n}$. Note that

$$\frac{\|A^k\|}{k^r \rho(A)^k} = \frac{\|A^k\|}{k^r \delta^k}$$

is bounded with respect to $k > 0$.

Let $C_1 = \sup_{k>0} \{\|A^k\|/k^r \delta^k\}$. Set $C_2 = \max\{1, C_1, CC_1, C\}$. Then dividing inequality (8) by $n^r \delta^n$, we get

$$\begin{aligned}
 (9) \quad & \frac{h_H(f^n(P))}{n^r \delta^n} \\
 & \leq C \left(\frac{\|A^{n-1}\|}{n^r \delta^n} \sqrt{h_H(P)} + \right. \\
 & \quad \sum_{k=1}^{n-2} \frac{\|A^{n-1-k}\|}{(n-1-k)^r \delta^{n-1-k}} \sqrt{\frac{h_H(f^k(P))}{k^r \delta^k}} \frac{(n-1-k)^r k^{r/2}}{n^r \delta^{1+k/2}} \\
 & \quad \left. + \sqrt{\frac{h_H(f^{n-1}(P))}{(n-1)^r \delta^{n-1}}} \frac{(n-1)^{r/2}}{n^r \delta^{1+(n-1)/2}} \right) + \frac{\|A^n\|}{n^r \delta^n} h_H(P) \\
 & \leq C_2 \left(\sqrt{h_H(P)} + \sum_{k=1}^{n-2} \sqrt{\frac{h_H(f^k(P))}{k^r \delta^k}} \frac{(n-1-k)^r k^{r/2}}{n^r \delta^{1+k/2}} \right. \\
 & \quad \left. + \sqrt{\frac{h_H(f^{n-1}(P))}{(n-1)^r \delta^{n-1}}} \frac{(n-1)^{r/2}}{n^r \delta^{1+(n-1)/2}} + h_H(P) \right).
 \end{aligned}$$

First we assume that $\delta > 1$. Then $k^{r/2}/\delta^{1+k/2}$ is bounded with respect to k , there exists a constant $C_3 > 0$ which is independent of n, P and

$$\frac{h_H(f^n(P))}{n^r \delta^n} \leq C_3 \left(\sqrt{h_H(P)} + \sum_{k=1}^{n-1} \sqrt{\frac{h_H(f^k(P))}{k^r \delta^k}} + h_H(P) \right).$$

Applying Lemma A.2 to $a_n = h_H(f^n(P))/n^r \delta^n$, there exists a constant $C_4 > 0$ independent of n, P such that

$$\frac{h_H(f^n(P))}{n^r \delta^n} \leq C_4 n^2 h_H(P).$$

Again from (9),

$$\frac{h_H(f^n(P))}{n^r \delta^n} \leq C_2 \left(\sqrt{h_H(P)} + \sum_{k=1}^{n-1} \sqrt{C_4 h_H(P)} \frac{k^{1+r/2}}{\delta^{1+k/2}} + h_H(P) \right).$$

Since $\sum_{k=1}^{\infty} k^{1+r/2}/\delta^{1+k/2}$ is convergent, there exists a constant $C_5 > 0$ independent of n, P such that

$$\frac{h_H(f^n(P))}{n^r \delta^n} \leq C_5 h_H(P).$$

Thus $h_H(f^n(P)) \leq C_5 n^r \delta^n h_H(P)$. Now, since h_H and h_X are ample height functions and we take $h_H \geq 1$, there exists an integer $m > 0$ such that

$$mh_H \geq h_X^+, \quad mh_X^+ \geq h_H.$$

Thus

$$h_X^+(f^n(P)) \leq mh_H(f^n(P)) \leq mC_5 n^r \delta^n h_H(P) \leq m^2 C_5 n^r \delta^n h_X^+(P).$$

This completes the proof of Theorem 2.1(2).

Now assume that $\delta = 1$. Dividing both sides of (9) by n^r , we get

$$\begin{aligned} \frac{h_H(f^n(P))}{n^{2r}} &\leq C_2 \left(\frac{\sqrt{h_H(P)}}{n^r} + \sum_{k=1}^{n-2} \sqrt{\frac{h_H(f^k(P))}{k^{2r}}} \frac{(n-1-k)^r k^r}{n^{2r}} \right. \\ &\quad \left. + \sqrt{\frac{h_H(f^{n-1}(P))}{(n-1)^{2r}}} \frac{(n-1)^r}{n^{2r}} + \frac{h_H(P)}{n^r} \right) \\ &\leq C_2 \left(\sqrt{h_H(P)} + \sum_{k=1}^{n-1} \sqrt{\frac{h_H(f^k(P))}{k^{2r}}} + h_H(P) \right). \end{aligned}$$

By Lemma A.2, there exists a constant $C_6 > 0$ independent of n, P such that

$$h_H(f^n(P)) \leq C_6 n^{2r+2} h_H(P) \quad \text{for all } n \geq 1.$$

By the same argument at the end of the proof of (2), this proves Theorem 2.1(1). \square

3. RATIONAL SELF-MAP CASE

Now we prove the main theorem of this paper.

Theorem 3.1 (Theorem 1.4). *Let X be a smooth projective variety over $\overline{\mathbb{Q}}$ and $f : X \dashrightarrow X$ be a dominant rational map defined over $\overline{\mathbb{Q}}$. Let δ_f be the first dynamical degree of f . Fix an ample height function h_X on X . For any $\epsilon > 0$, there exists $C > 0$ such that*

$$h_X^+(f^n(P)) \leq C(\delta_f + \epsilon)^n h_X^+(P)$$

for all $n \geq 0$ and $P \in X_f(\overline{\mathbb{Q}})$. In particular, for any $P \in X_f(\overline{\mathbb{Q}})$, we have

$$\overline{\alpha}_f(P) \leq \delta_f.$$

We deduce this theorem from the following theorem.

Theorem 3.2. *Let X be a smooth projective variety over $\overline{\mathbb{Q}}$ and $f : X \dashrightarrow X$ be a dominant rational map defined over $\overline{\mathbb{Q}}$ with first dynamical degree δ_f . Fix an ample height function h_X on X . Then, for any $\epsilon > 0$, there exist a positive integer k and a constant $C > 0$ such that*

$$h_X^+(f^{nk}(P)) \leq C(\delta_f + \epsilon)^{nk} h_X^+(P)$$

for all $n \geq 0$ and $P \in X_f(\overline{\mathbb{Q}})$.

Lemma 3.3. *In the situation of Theorem 3.2, there exists a constant $C_0 \geq 1$ such that*

$$h_X^+(f^n(P)) \leq C_0^n h_X^+(P)$$

for all $n \geq 0$ and $P \in X_f(\overline{\mathbb{Q}})$.

Proof. Let H be an ample divisor on X . Take a height function h_H associated with H so that $h_H \geq 1$. Let h_{f^*H} be a height function associated with f^*H . Then, from [9, Proposition 21]

$$h_H(f(P)) \leq h_{f^*H}(P) + O(1)$$

for all $P \in X_f(\overline{\mathbb{Q}})$. Here $O(1)$ is a bounded function on $X_f(\overline{\mathbb{Q}})$ which depends on $f, H, f^*H, h_H, h_{f^*H}$ but is independent of P . Since H is ample and $h_H \geq 1$, for a sufficiently large $C_0 \geq 1$, we have

$$h_{f^*H}(P) + O(1) \leq C_0 h_H(P)$$

for all $P \in X_f(\overline{\mathbb{Q}})$. Thus, we get

$$h_H(f(P)) \leq C_0 h_H(P)$$

for all $P \in X_f(\overline{\mathbb{Q}})$. Therefore

$$h_H(f^n(P)) \leq C_0^n h_H(P).$$

By Remark 1.12 or the same argument at the end of the proof of Theorem 2.1(2), this proves the statement. \square

Proof of Theorem 3.2 \implies Theorem 3.1. From Theorem 3.2, for any $\epsilon > 0$, there exist a positive integer k and a positive constant $C > 0$ such that

$$h_X^+(f^{nk}(P)) \leq C(\delta_f + \epsilon)^{nk} h_X^+(P)$$

for all $n \geq 0$ and $P \in X_f(\overline{\mathbb{Q}})$. For any integer $m \geq 0$, we write $m = qk + t$ $q \geq 0, 0 \leq t < k$. Let C_0 be the constant in Lemma 3.3. Then for any $P \in X_f(\overline{\mathbb{Q}})$,

$$\begin{aligned} h_X^+(f^m(P)) &\leq C(\delta_f + \epsilon)^{qk} h_X^+(f^t(P)) \\ &\leq C C_0^t (\delta_f + \epsilon)^{qk} h_X^+(P) \\ &\leq C C_0^{k-1} (\delta_f + \epsilon)^m h_X^+(P). \end{aligned}$$

This proves the first statement in Theorem 3.1.

The second statement is an easy consequence of the first one. That is,

$$\begin{aligned} \overline{\alpha}_f(P) &= \limsup_{n \rightarrow \infty} h_X^+(f^n(P))^{1/n} \\ &\leq \limsup_{n \rightarrow \infty} (C h_X^+(P))^{1/n} (\delta_f + \epsilon) \\ &= \delta_f + \epsilon. \end{aligned}$$

Since ϵ is arbitrary, we get $\overline{\alpha}_f(P) \leq \delta_f$. \square

Before starting the proof of Theorem 3.2, we prove an interesting corollary.

Corollary 3.4. *In the situation of Theorem 3.2,*

$$\overline{\alpha}_f(P) = \limsup_{n \rightarrow \infty} h_X^+(f^{nk}(P))^{1/nk} = \overline{\alpha}_{f^k}(P)^{1/k}$$

for any $k > 0$ and any point $P \in X_f(\overline{\mathbb{Q}})$.

Proof.

$$\begin{aligned} \overline{\alpha}_f(P) &= \limsup_{m \rightarrow \infty} h_X^+(f^m(P))^{1/m} \\ &= \limsup_{n \rightarrow \infty} \max_{0 \leq i < k} h_X^+(f^{nk+i}(P))^{1/nk+i} \\ &\leq \limsup_{n \rightarrow \infty} \max_{0 \leq i < k} (C_0^i h_X^+(f^{nk}(P)))^{1/nk+i} && \text{by Lemma 3.3} \\ &\leq \limsup_{n \rightarrow \infty} (C_0^{k-1} h_X^+(f^{nk}(P)))^{1/nk} \\ &= \limsup_{n \rightarrow \infty} h_X^+(f^{nk}(P))^{1/nk} \\ &\leq \overline{\alpha}_{f^k}(P). \end{aligned}$$

\square

Now we turn to the proof of Theorem 3.2.

Proof of Theorem 3.2. Let D_1, \dots, D_r be very ample divisors on X which forms a basis for $N^1(X)_{\mathbb{R}}$. Take an ample divisor H on X so that $H \pm D_i$, $i = 1, \dots, r$ are ample and if we write $H \equiv \sum_{i=1}^r c_i D_i$ then $c_i \geq 0$.

We take a resolution of indeterminacy $p : Y \rightarrow X$ of f as follows. p is a sequence of blowing ups at smooth centers and the images of centers in X are contained in the indeterminacy locus I_f of f . Let $g = f \circ p$.

$$\begin{array}{ccc} & Y & \\ p \swarrow & & \searrow g \\ X & \xrightarrow{\quad f \quad} & X \end{array}$$

Let $\text{Exc}(p)$ be the exceptional locus of p . Take general effective divisors $\widetilde{D}_1, \dots, \widetilde{D}_r$ on X such that \widetilde{D}_i is linearly equivalent to D_i and any components of $g^* \widetilde{D}_i$ are not contained in $\text{Exc}(p)$. Then

$$Z_i = p^* p_* g^* \widetilde{D}_i - g^* \widetilde{D}_i$$

is an effective divisor on Y whose support is contained in $\text{Exc}(p)$. Let $F_i = g^* \widetilde{D}_i$ for $i = 1, \dots, r$. Then $p^* p_* F_i - F_i$ and Z_i are linearly equivalent.

$$(10) \quad p^* p_* F_i - F_i \sim_{\text{lin}} Z_i$$

Take divisors F_{r+1}, \dots, F_s on Y so that F_1, \dots, F_s forms a basis for $N^1(Y)_{\mathbb{R}}$. There exists an ample \mathbb{Q} -divisor H' on Y such that $p^* H - H'$ is an effective divisor whose support is contained in $\text{Exc}(p)$. (see for example [5, Proposition 7.10].) Let

$$(11) \quad g^* D_i \equiv \sum_{m=1}^s a_{mi} F_m \quad (i = 1, \dots, r)$$

$$(12) \quad p_* F_j \equiv \sum_{l=1}^r b_{lj} D_l \quad (j = 1, \dots, s)$$

and

$$A = (a_{mi})_{mi} \quad s \times r\text{-matrix}$$

$$B = (b_{lj})_{lj} \quad r \times s\text{-matrix}.$$

By the definition of F_j , A is the following form.

$$(13) \quad A = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}.$$

Note that BA is the representation matrix of f^* with respect to the basis D_1, \dots, D_r . We write

$$\vec{D} = \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_r \end{pmatrix}, \vec{F} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_s \end{pmatrix}, \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{pmatrix}, \vec{Z} = \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_r \end{pmatrix}.$$

Let

$$(14) \quad E = g^*H - \langle A\vec{c}, \vec{F} \rangle$$

$$(15) \quad \vec{E}' = \begin{pmatrix} E'_1 \\ E'_2 \\ \vdots \\ E'_s \end{pmatrix} = p_*\vec{F} - {}^t B \vec{D}.$$

These are numerically zero divisors.

The choice of height functions.

Fix height functions h_{D_1}, \dots, h_{D_r} associated with D_1, \dots, D_r . Fix a height function h_H associated with H so that $h_H \geq 1$ and $h_H \geq |h_{D_i}|$ for $i = 1, \dots, r$. Note that h_{D_1}, \dots, h_{D_r} and h_H are independent of f .

We define $h_{F_j} = h_{D_j} \circ g$, $j = 1, \dots, r$. These are height functions associated with F_j . For $j = r+1, \dots, s$, fix any height functions h_{F_j} associated with F_j . Fix height functions $h_{p_*F_j}$ associated with p_*F_j for $j = 1, \dots, s$. We write

$$\mathbf{h}_{\vec{D}} = \begin{pmatrix} h_{D_1} \\ h_{D_2} \\ \vdots \\ h_{D_r} \end{pmatrix}, \quad \mathbf{h}_{\vec{F}} = \begin{pmatrix} h_{F_1} \\ h_{F_2} \\ \vdots \\ h_{F_s} \end{pmatrix}, \quad \mathbf{h}_{p_*\vec{F}} = \begin{pmatrix} h_{p_*F_1} \\ h_{p_*F_2} \\ \vdots \\ h_{p_*F_s} \end{pmatrix}.$$

Define

$$(16) \quad \mathbf{h}_{\vec{E}'} = \begin{pmatrix} h_{E'_1} \\ h_{E'_2} \\ \vdots \\ h_{E'_s} \end{pmatrix} = \mathbf{h}_{p_*\vec{F}} - {}^t B \mathbf{h}_{\vec{D}}$$

$$(17) \quad \mathbf{h}_{\vec{Z}} = \begin{pmatrix} h_{Z_1} \\ h_{Z_2} \\ \vdots \\ h_{Z_r} \end{pmatrix} = \begin{pmatrix} h_{p_*F_1} \\ h_{p_*F_2} \\ \vdots \\ h_{p_*F_r} \end{pmatrix} \circ p - \begin{pmatrix} h_{F_1} \\ h_{F_2} \\ \vdots \\ h_{F_r} \end{pmatrix}.$$

By (15) and (10), $h_{E'_j}$ is a height function associated with E'_j for $j = 1, \dots, s$ and h_{Z_i} is a height function associated with Z_i for $i = 1, \dots, r$. By adding a bounded function to $h_{p_*F_i}$, we may assume that $h_{Z_i} \geq 0$ on $Y \setminus Z_i$. Fix a height function $h_{H'} \geq 1$ associated with H' . Fix a height function $h_{p^*H-H'}$ associated with p^*H-H' so that $h_{p^*H-H'} \geq 0$ on $Y \setminus \text{Exc}(p)$. Note that there exists a constant $\gamma \geq 0$ such that

$$(18) \quad h_H \circ p \geq h_{p^*H-H'} + h_{H'} - \gamma \quad \text{on } Y(\overline{\mathbb{Q}}).$$

Finally we define

$$(19) \quad h_E = h_H \circ g - \langle A\vec{c}, \mathbf{h}_{\vec{F}} \rangle.$$

By (14), this is a height function associated with E . Since E, E'_j are numerically zero, there exists a constant $C > 0$ such that

$$(20) \quad |h_E| \leq C\sqrt{h_{H'}}$$

$$(21) \quad |h_{E'_j}| \leq C\sqrt{h_H}.$$

Let $M(f)$ be the representation matrix of the linear map $f^* : N^1(X)_{\mathbb{R}} \rightarrow N^1(X)_{\mathbb{R}}$ with respect to the basis D_1, \dots, D_r .

Claim. Let $R = \max\{1, r^2 \|\vec{c}\| \|M(f)\|\}$. Then there exists $K > 0$ such that

$$h_H(f^n(P)) \leq K n^2 R^n h_H(P)$$

for all $n \geq 1$ and $P \in X_f(\overline{\mathbb{Q}})$. Note that the constant K depends on f but h_H, r, \vec{c} and D_1, \dots, D_r do not depend on f .

Proof of the claim. Let $P \in X_f(\overline{\mathbb{Q}})$. Let $\|\cdot\|$ be the max norm of matrices. For $n \geq 1$

$$\begin{aligned}
(22) \quad & h_H(f^n(P)) \\
&= (h_H \circ g)(p^{-1} f^{n-1}(P)) \\
&\quad - \left\langle A\vec{c}, \mathbf{h}_{p_* \vec{F}} \circ p \right\rangle (p^{-1} f^{n-1}(P)) \\
&\quad + \left\langle A\vec{c}, \mathbf{h}_{p_* \vec{F}} \right\rangle (f^{n-1}(P)) \\
&= \left\langle A\vec{c}, \mathbf{h}_{\vec{F}} - \mathbf{h}_{p_* \vec{F}} \circ p \right\rangle (p^{-1} f^{n-1}(P)) \\
&\quad + h_E(p^{-1} f^{n-1}(P)) \\
&\quad + \left\langle BA\vec{c}, \mathbf{h}_{\vec{D}} \right\rangle (f^{n-1}(P)) \\
&\quad + \left\langle A\vec{c}, \mathbf{h}_{\vec{E}'} \right\rangle (f^{n-1}(P)) \quad \text{by (16)(19)} \\
&= \left\langle \vec{c}, -\mathbf{h}_{\vec{Z}} \right\rangle (p^{-1} f^{n-1}(P)) \\
&\quad + h_E(p^{-1} f^{n-1}(P)) \\
&\quad + \left\langle BA\vec{c}, \mathbf{h}_{\vec{D}} \right\rangle (f^{n-1}(P)) \\
&\quad + \left\langle \vec{c}, {}^t A \mathbf{h}_{\vec{E}'} \right\rangle (f^{n-1}(P)) \quad \text{by (17)} \\
&\leq h_E(p^{-1} f^{n-1}(P)) \\
&\quad + \left\langle BA\vec{c}, \mathbf{h}_{\vec{D}} \right\rangle (f^{n-1}(P)) \quad \text{since } h_{Z_i} \geq 0 \text{ on } Y \setminus \text{Exc}(p) \\
&\quad + \left\langle \vec{c}, {}^t A \mathbf{h}_{\vec{E}'} \right\rangle (f^{n-1}(P)) \\
&\leq r^2 \|\vec{c}\| \|BA\| h_H(f^{n-1}(P)) \\
&\quad + r \|\vec{c}\| C \sqrt{h_H(f^{n-1}(P))} \quad \text{by (13)(20)(21)} \\
&\quad + C \sqrt{h_{H'}(p^{-1}(f^{n-1}(P)))} \\
&\leq r^2 \|\vec{c}\| \|BA\| h_H(f^{n-1}(P)) \quad \text{by (18) and} \\
&\quad + r \|\vec{c}\| C \sqrt{h_H(f^{n-1}(P))} \quad h_{p^* H - H'} \geq 0 \text{ on } Y \setminus \text{Exc}(p) \\
&\quad + C \sqrt{h_H(f^{n-1}(P))} + \gamma.
\end{aligned}$$

Note that C, γ depend on f . On the other hand, r, H, D_1, \dots, D_r , and h_H do not depend on f . Thus \vec{c} also does not depend on f .

Since BA is the representation matrix of f^* with respect to D_1, \dots, D_r , $BA = M(f)$ and $R = \max\{1, r^2 \|\vec{c}\| \|BA\|\}$. Then, dividing the both sides of (22) by R^n ,

we get

$$\begin{aligned} \frac{h_H(f^n(P))}{R^n} &\leq \frac{h_H(f^{n-1}(P))}{R^{n-1}} \\ &\quad + r\|\bar{c}\|C\sqrt{\frac{h_H(f^{n-1}(P))}{R^{n-1}}} + C\sqrt{\frac{h_H(f^{n-1}(P))}{R^{n-1}}} + \gamma. \end{aligned}$$

Let

$$a_n = \frac{h_H(f^n(P))}{R^n} \quad \text{for } n \geq 0.$$

Then $a_n > 0$ and $a_0 = h_H(P)$ and the sequence $(a_n)_n$ satisfies the following inequality.

$$a_n \leq a_{n-1} + r\|\bar{c}\|C\sqrt{a_{n-1}} + C\sqrt{a_{n-1}} + \gamma$$

By Lemma A.1, there exist a constant $K > 0$ independent of n, P such that

$$a_n \leq Kn^2 a_0 \quad \text{for all } n \geq 1.$$

Therefore

$$h_H(f^n(P)) \leq Kn^2 R^n h_H(P).$$

Thus we get the claim. \square

Now, fix any positive real number $\epsilon > 0$. Let $\delta = \delta_f$. Let $M(f^k)$ be the representation matrix of $(f^k)^* : N^1(X)_{\mathbb{R}} \rightarrow N^1(X)_{\mathbb{R}}$ with respect to the basis D_1, \dots, D_r . Since $\lim_{k \rightarrow \infty} \|M(f^k)\|^{1/k} = \delta$, there exists a positive integer $k > 0$ such that

$$(23) \quad \frac{\|M(f^k)\|}{(\delta + \epsilon)^k} r^2 \|\bar{c}\| < 1.$$

Fix such a k and we apply the claim to f^k . Then,

$$h_H(f^{kn}(P)) \leq Kn^2 \left(\frac{R}{(\delta + \epsilon)^k} \right)^n (\delta + \epsilon)^{kn} h_H(P).$$

Recall $R = \max\{1, r^2 \|\bar{c}\| \|M(f^k)\|\}$. Thus, by (23)

$$\frac{R}{(\delta + \epsilon)^k} < 1.$$

Thus there exists a constant K' such that

$$Kn^2 \left(\frac{R}{(\delta + \epsilon)^k} \right)^n \leq K'$$

for all n . Then we get

$$h_H(f^{kn}(P)) \leq K'(\delta + \epsilon)^{kn} h_H(P).$$

By Remark 1.12 or the same argument at the end of the proof of Theorem 2.1(2), this proves Theorem 3.2(2). \square

Remark 3.5. One can prove Theorem 3.1 over any ground field K such that Weil height functions can be defined. If the characteristic of K is zero, the same proof is valid. For the case when the characteristic of K is positive, see Appendix B.

4. PICARD RANK ONE CASE

When the Picard rank of X is one, we can prove different type inequalities.

Theorem 4.1 (Theorem 1.8). *Let X be a smooth projective variety over $\overline{\mathbb{Q}}$ of Picard number one. Let $f : X \dashrightarrow X$ be a dominant rational self-map defined over $\overline{\mathbb{Q}}$. Fix an ample height function h_X on X .*

- (1) *For any positive integer $k > 0$, there exists a constant $C > 0$ such that*

$$h_X^+(f^n(P)) \leq Cn^2 \rho((f^k)^*)^{n/k} h_X^+(P)$$

for all $P \in X_f(\overline{\mathbb{Q}})$ and $n \geq 1$.

- (2) *Let $k > 0$ be a positive integer. Assume that $\rho((f^k)^*) > 1$. Then there exists a constant $C > 0$ such that*

$$h_X^+(f^n(P)) \leq C \rho((f^k)^*)^{n/k} h_X^+(P)$$

for all $P \in X_f(\overline{\mathbb{Q}})$ and $n \geq 0$.

Proof. We use the notation in the proof of Theorem 3.2. For simplicity, we write $\rho_k = \rho((f^k)^*)$ for $k > 0$. We apply (22) to f^k . By the assumption $r = 1$, thus $BA = \rho_k$ is a real number. By (22),

$$\begin{aligned} (24) \quad h_H(f^{nk}(P)) &= -c_1 h_{Z_1}(p^{-1} f^{k(n-1)}(P)) + h_E(p^{-1} f^{k(n-1)}(P)) \\ &\quad + \rho_k c_1 h_{D_1}(f^{k(n-1)}(P)) + c_1 h_{E'_1}(f^{k(n-1)}(P)) \\ &\leq \rho_k c_1 h_{D_1}(f^{k(n-1)}(P)) + C \sqrt{h_H(f^{k(n-1)}(P))} + \gamma \\ &\quad + c_1 C \sqrt{h_H(f^{k(n-1)}(P))} \end{aligned}$$

Let $N = c_1 D_1 - H$. By the definition of c_1 , this is a numerically zero divisor. Define

$$h_N = c_1 h_{D_1} - h_H.$$

Then, this is a height function associated with N . Thus there exists a constant $\tilde{C} > 0$ such that

$$|h_N| \leq \tilde{C} \sqrt{h_H}.$$

Then

$$\begin{aligned} h_H(f^{nk}(P)) &\leq \rho_k h_H(f^{k(n-1)}(P)) + \tilde{C} \sqrt{h_H(f^{k(n-1)}(P))} \\ &\quad + C \sqrt{h_H(f^{k(n-1)}(P))} + \gamma + c_1 C \sqrt{h_H(f^{k(n-1)}(P))}. \end{aligned}$$

Divide both sides of this inequality by ρ_k^n . By Lemma A.1, there exists a constant $\tilde{K} > 0$ (which is independent of n, P , but depends on k) such that

$$(25) \quad h_H(f^{nk}(P)) \leq \tilde{K} n^2 \rho_k^{nk/k} h_H(P) \quad \text{for all } n \geq 1.$$

By the same argument as in (Proof of Theorem 3.2 \implies Theorem 3.1), we can prove the first statement.

Now assume $\rho_k > 1$. Then

$$\frac{h_H(f^{nk}(P))}{\rho_k^n} \leq \frac{h_H(f^{k(n-1)}(P))}{\rho_k^{n-1}} + (\tilde{C} + C + c_1 C) \frac{\sqrt{h_H(f^{k(n-1)}(P))}}{\rho_k^n} + \frac{C\sqrt{\gamma}}{\rho_k^n}$$

By (25),

$$\sqrt{h_H(f^{k(n-1)}(P))} \leq \sqrt{\tilde{K} h_H(P) (n-1) \rho_k^{(n-1)/2}}$$

and thus

$$\begin{aligned} & \sum_{n=1}^{\infty} \left\{ (\tilde{C} + C + c_1 C) \frac{\sqrt{h_H(f^{k(n-1)}(P))}}{\rho_k^n} + \frac{C\sqrt{\gamma}}{\rho_k^n} \right\} \\ & \leq \sum_{n=1}^{\infty} \left\{ (\tilde{C} + C + c_1 C) \frac{\sqrt{\tilde{K} h_H(P)(n-1)\rho_k^{(n-1)/2}}}{\rho_k^n} + \frac{C\sqrt{\gamma}}{\rho_k^n} \right\}. \end{aligned}$$

Since $\rho_k > 1$, there exists a constant \tilde{K}_1 (independent of n, P) such that

$$\frac{h_H(f^{nk}(P))}{\rho_k^n} \leq \tilde{K}_1 h_H(P).$$

Thus

$$h_H(f^{nk}(P)) \leq \tilde{K}_1 \rho_k^{nk/k} h_H(P).$$

By the same argument as in (Proof of Theorem 3.2 \implies Theorem 3.1), we can prove the second statement. \square

Finally we prove Proposition 1.9.

Proposition 4.2 (Proposition 1.9). *Let X and f be as in Theorem 4.1. Assume f is algebraically stable and $\delta_f > 1$. Fix an ample height function h_X on X . Then*

$$\hat{h}_{X,f}(P) = \lim_{n \rightarrow \infty} \frac{h_X(f^n(P))}{\delta_f^n}$$

exists for all $P \in X_f(\overline{\mathbb{Q}})$.

Proof. For simplicity, we write $\delta = \delta_f$. We use the notation in the proof of Theorem 3.2 and Theorem 4.1. Note that $\delta = \rho_1$ since f is assumed algebraically stable. By (24), for any $P \in X_f(\overline{\mathbb{Q}})$ and $n \geq 1$, we have

$$\begin{aligned} h_H(f^n(P)) &= - \sum_{i=0}^{n-1} \delta^{n-1-i} c_1 h_{Z_1}(p^{-1}(f^i(P))) + \sum_{i=0}^{n-1} \delta^{n-1-i} h_E(p^{-1}(f^i(P))) \\ &\quad + \sum_{i=0}^{n-1} \delta^{n-1-i} c_1 h_{E'_1}(f^i(P)) + \sum_{i=0}^{n-1} \delta^{n-1-i} h_N(f^i(P)) + \delta^n h_H(P). \end{aligned}$$

Dividing by δ^n , we get

$$\begin{aligned} \frac{h_H(f^n(P))}{\delta^n} &= - \sum_{i=0}^{n-1} \delta^{-1-i} c_1 h_{Z_1}(p^{-1}(f^i(P))) + \sum_{i=0}^{n-1} \delta^{-1-i} h_E(p^{-1}(f^i(P))) \\ &\quad + \sum_{i=0}^{n-1} \delta^{-1-i} c_1 h_{E'_1}(f^i(P)) + \sum_{i=0}^{n-1} \delta^{-1-i} h_N(f^i(P)) + h_H(P). \end{aligned}$$

By Theorem 4.1, when $n \rightarrow \infty$, the second, third and fourth terms are convergent (note that E, E'_1, N are numerically zero). By the construction of c_1 and h_{Z_1} , $c_1 h_{Z_1}(p^{-1}(f^k(P))) \geq 0$. Since $h_H(f^n(P))/\delta^n \geq 0$, the first term is also convergent. Hence the limit

$$\lim_{n \rightarrow \infty} \frac{h_H(f^n(P))}{\delta^n}$$

exists. Since the Picard rank of X is one, there exist $a > 0$ and $C > 0$ such that

$$|h_X - ah_H| \leq C\sqrt{h_H}.$$

By Theorem 4.1(2), $\lim_{n \rightarrow \infty} \sqrt{h_H(f^n(P))}/\delta^n = 0$. Thus the limit

$$\lim_{n \rightarrow \infty} \frac{h_X(f^n(P))}{\delta^n}$$

exists. □

With a few more effort, one can prove a more general statement as follows.

Proposition 4.3. *Let X be a smooth projective variety over $\overline{\mathbb{Q}}$. Let $f : X \dashrightarrow X$ be a dominant rational self-map defined over $\overline{\mathbb{Q}}$. Assume $\delta_f > 1$ and there exists a nef \mathbb{R} -divisor H on X such that $f^*H \equiv \delta_f H$. Fix a height function h_H associated with H . Then for any $P \in X_f(\overline{\mathbb{Q}})$, if the sequence $(h_H(f^n(P))/\delta_f^n)_{n \geq 0}$ is bounded below, the limit*

$$\lim_{n \rightarrow \infty} \frac{h_H(f^n(P))}{\delta_f^n}$$

exists.

Proof. Although the argument is very similar to the proof of Theorem 3.2, we give the proof for the benefit of the reader.

Let $D_1 = H$ and D_2, \dots, D_r be very ample divisors on X such that D_1, \dots, D_r form a basis for $N^1(X)_{\mathbb{R}}$.

We take a resolution of indeterminacy $p : Y \rightarrow X$ of f as follows. p is a sequence of blowing ups at smooth centers and the images of centers in X are contained in the indeterminacy locus I_f of f . Let $g = f \circ p$.

$$\begin{array}{ccc} & Y & \\ p \swarrow & & \searrow g \\ X & \dashrightarrow f \dashrightarrow & X \end{array}$$

Let $\text{Exc}(p)$ be the exceptional locus of p . Then, by Negativity lemma

$$Z_i = p^*p_*g^*D_i - g^*D_i$$

is an effective divisor on Y whose support is contained in $\text{Exc}(p)$. Let $F_i = g^*D_i$ for $i = 1, \dots, r$. Take divisors F_{r+1}, \dots, F_s on Y so that F_1, \dots, F_s forms a basis for $N^1(Y)_{\mathbb{R}}$.

Let

$$(26) \quad g^*D_i \equiv \sum_{m=1}^s a_{mi} F_m \quad (i = 1, \dots, r)$$

$$(27) \quad p_*F_j \equiv \sum_{l=1}^r b_{lj} D_l \quad (j = 1, \dots, s)$$

and

$$A = (a_{mi})_{mi} \quad s \times r\text{-matrix}$$

$$B = (b_{lj})_{lj} \quad r \times s\text{-matrix}.$$

By the definition of F_j , A is the following form.

$$(28) \quad A = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}.$$

Note that BA is the representation matrix of f^* with respect to the basis D_1, \dots, D_r . We write

$$\vec{D} = \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_r \end{pmatrix}, \vec{F} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_s \end{pmatrix}, \vec{e} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \vec{Z} = \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_r \end{pmatrix}.$$

Since $D_1 = H$, we have $BA\vec{e} = \delta_f \vec{e}$. Let

$$(29) \quad \vec{E}' = \begin{pmatrix} E'_1 \\ E'_2 \\ \vdots \\ E'_s \end{pmatrix} = p_* \vec{F} - {}^t B \vec{D}.$$

These are numerically zero divisors.

The choice of height functions.

Fix height functions h_{D_1}, \dots, h_{D_r} associated with D_1, \dots, D_r .

We define $h_{F_j} = h_{D_j} \circ g$, $j = 1, \dots, r$. These are height functions associated with F_j . For $j = r+1, \dots, s$, fix any height functions h_{F_j} associated with F_j . Fix height functions $h_{p_* F_j}$ associated with $p_* F_j$ for $j = 1, \dots, s$. We write

$$\mathbf{h}_{\vec{D}} = \begin{pmatrix} h_{D_1} \\ h_{D_2} \\ \vdots \\ h_{D_r} \end{pmatrix}, \mathbf{h}_{\vec{F}} = \begin{pmatrix} h_{F_1} \\ h_{F_2} \\ \vdots \\ h_{F_s} \end{pmatrix}, \mathbf{h}_{p_* \vec{F}} = \begin{pmatrix} h_{p_* F_1} \\ h_{p_* F_2} \\ \vdots \\ h_{p_* F_s} \end{pmatrix}.$$

Define

$$(30) \quad \mathbf{h}_{\vec{E}'} = \begin{pmatrix} h_{E'_1} \\ h_{E'_2} \\ \vdots \\ h_{E'_s} \end{pmatrix} = \mathbf{h}_{p_* \vec{F}} - {}^t B \mathbf{h}_{\vec{D}}$$

$$(31) \quad \mathbf{h}_{\vec{Z}} = \begin{pmatrix} h_{Z_1} \\ h_{Z_2} \\ \vdots \\ h_{Z_r} \end{pmatrix} = \begin{pmatrix} h_{p_* F_1} \\ h_{p_* F_2} \\ \vdots \\ h_{p_* F_r} \end{pmatrix} \circ p - \begin{pmatrix} h_{F_1} \\ h_{F_2} \\ \vdots \\ h_{F_r} \end{pmatrix}.$$

By the definition of E'_j and Z_j , $h_{E'_j}$ is a height function associated with E'_j for $j = 1, \dots, s$ and h_{Z_i} is a height function associated with Z_i for $i = 1, \dots, r$. By adding a bounded function to $h_{p_* F_i}$, we may assume that $h_{Z_i} \geq 0$ on $Y \setminus Z_i$.

Let $P \in X_f(\overline{\mathbb{Q}})$. For $n \geq 1$

$$h_H(f^n(P))$$

$$\begin{aligned}
&= (h_H \circ g)(p^{-1}f^{n-1}(P)) \\
&\quad - \langle A\vec{e}, \mathbf{h}_{p_*\vec{F}} \circ p \rangle (p^{-1}f^{n-1}(P)) \\
&\quad + \langle A\vec{e}, \mathbf{h}_{p_*\vec{F}} \rangle (f^{n-1}(P)) \\
&= \langle A\vec{e}, \mathbf{h}_{\vec{F}} - \mathbf{h}_{p_*\vec{F}} \circ p \rangle (p^{-1}f^{n-1}(P)) \\
&\quad + \langle A\vec{e}, {}^tB\mathbf{h}_{\vec{D}} + \mathbf{h}_{\vec{E}'} \rangle (f^{n-1}(P)) \\
&= \langle A\vec{e}, -\mathbf{h}_{\vec{Z}} \rangle (p^{-1}f^{n-1}(P)) \\
&\quad + \langle BA\vec{e}, \mathbf{h}_{\vec{D}} \rangle (f^{n-1}(P)) + \langle A\vec{e}, \mathbf{h}_{\vec{E}'} \rangle (f^{n-1}(P)) \\
&= -h_{Z_1}(p^{-1}f^{n-1}(P)) + \delta_f h_H(f^{n-1}(P)) + h_{E'_1}(f^{n-1}(P)).
\end{aligned}$$

Thus we get

$$h_H(f^n(P)) = \delta_f^n h_H(P) + \sum_{k=0}^{n-1} \delta_f^{n-1-k} h_{E'_1}(f^k(P)) - \sum_{k=0}^{n-1} \delta_f^{n-1-k} h_{Z_1}(p^{-1}f^k(P)).$$

Dividing by δ_f^n , we get

$$\frac{h_H(f^n(P))}{\delta_f^n} = h_H(P) + \sum_{k=0}^{n-1} \frac{h_{E'_1}(f^k(P))}{\delta_f^{k+1}} - \sum_{k=0}^{n-1} \frac{h_{Z_1}(p^{-1}f^k(P))}{\delta_f^{k+1}}.$$

Since E'_1 is numerically zero, the limit $\lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} h_{E'_1}(f^k(P))/\delta_f^{k+1}$ exists. By the choice of h_{Z_1} , $h_{Z_1}(p^{-1}f^k(P)) \geq 0$ and the sequence $(\sum_{k=0}^{n-1} h_{Z_1}(p^{-1}f^k(P))/\delta_f^{k+1})_n$ is weakly increasing. By the assumption, the left hand side is bounded below and therefore the limit $\lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} h_{Z_1}(p^{-1}f^k(P))/\delta_f^{k+1}$ exists. \square

APPENDIX A. LEMMAS

Lemma A.1. *Let $(a_n)_{n \geq 0}$ be a sequence of positive real numbers with $a_0 \geq 1$ which satisfies*

$$a_n \leq a_{n-1} + C_1 \left(\sqrt{a_{n-1}} + \sqrt{a_{n-1} + C_2} \right)$$

for all $n \geq 1$. Here C_1, C_2 are non-negative constants. Then there exists a positive constant \tilde{C} depending only on C_1, C_2 such that

$$a_n \leq \tilde{C} n^2 a_0$$

for all $n \geq 1$.

Proof. Define a sequence $(b_n)_{n \geq 0}$ as follows.

$$\begin{aligned}
b_0 &= a_0 \\
b_n &= b_{n-1} + C_1 \left(\sqrt{b_{n-1}} + \sqrt{b_{n-1} + C_2} \right) \quad \text{for } n \geq 1.
\end{aligned}$$

Then we have $a_n \leq b_n$. By the definition, $(b_n)_{n \geq 0}$ is monotonically increasing. In particular, $b_n \geq 1$. Thus

$$b_n = b_{n-1} + C_1 \sqrt{b_{n-1}} \left(1 + \sqrt{1 + \frac{C_2}{b_{n-1}}} \right) \leq b_{n-1} + C_1 \left(1 + \sqrt{1 + C_2} \right) \sqrt{b_{n-1}}.$$

Let $C_3 = C_1 (1 + \sqrt{1 + C_2})$.

Define a sequence $(c_n)_{n \geq 0}$ as follows.

$$\begin{aligned} c_0 &= b_0 \\ c_n &= c_{n-1} + C_3\sqrt{c_{n-1}} \quad \text{for } n \geq 1. \end{aligned}$$

Then we have $b_n \leq c_n$. We take \tilde{C} so that

$$\tilde{C} \geq \max \left\{ \frac{C_3^2}{4}, 1 + C_3 \right\}.$$

It is enough to show that $c_n \leq \tilde{C}n^2c_0$ for $n \geq 1$. We prove this inequality by induction on n . For $n = 1$

$$c_1 = c_0 + C_3\sqrt{c_0} \leq (1 + C_3)c_0 \leq \tilde{C}c_0.$$

Assume $c_n \leq \tilde{C}n^2c_0$. Then

$$\begin{aligned} c_{n+1} &\leq \tilde{C}n^2c_0 + C_3\sqrt{\tilde{C}n^2c_0} \\ &\leq \tilde{C}n^2c_0 + 2\sqrt{\tilde{C}}\sqrt{\tilde{C}n^2c_0} \\ &= \tilde{C} \left(n^2 + \frac{2n}{\sqrt{c_0}} \right) c_0 \\ &\leq \tilde{C}(n^2 + 2n)c_0 \\ &\leq \tilde{C}(n+1)^2c_0. \end{aligned}$$

□

Lemma A.2. *Let $(a_n)_{n \geq 0}$ be a positive real sequence with $a_0 \geq 1$ which satisfies*

$$a_n \leq C(a_0 + \sqrt{a_0} + \sqrt{a_1} + \cdots + \sqrt{a_{n-1}}) \quad \text{for all } n \geq 1$$

where C is a positive constant. For any $\tilde{C} \geq 1$ such that $\tilde{C} \geq \max\{\frac{C^2}{4}, 2C\}$, we have

$$a_n \leq \tilde{C}n^2a_0 \quad \text{for all } n \geq 1.$$

Proof. Let $(b_n)_{n \geq 0}$ be a sequence such that

$$\begin{aligned} b_0 &= a_0 \\ b_n &= C \left(b_0 + \sqrt{b_0} + \cdots + \sqrt{b_{n-1}} \right) \quad \text{for all } n \geq 1. \end{aligned}$$

Then clearly $a_n \leq b_n$ for all $n \geq 0$. By the definition of b_n , we have $b_{n+1} = b_n + C\sqrt{b_n}$. Thus the statement follows from Lemma A.1. □

APPENDIX B. POSITIVE CHARACTERISTIC

In this section, we briefly remark how to arrange the proof of Theorem 3.2 when the ground field has positive characteristic. Let K be an algebraically closed field “with height function”.

Proposition B.1. *Let $f: X \dashrightarrow Z$ be a dominant rational map of smooth projective varieties over K .*

- (1) *Let Y be a projective variety with a birational morphism $p: Y \rightarrow X$ and a morphism $g: Y \rightarrow Z$ such that $f \circ p = g$. For a Cartier divisor D on Z , we define $f^*D = p_*[g^*D]$. Here $[g^*D]$ is the codimension one cycle associated with the Cartier divisor g^*D . This f^*D is independent of the choice of Y .*

- (2) Let $\Gamma \subset X \times Z$ be the graph of f . For a Cartier divisor D on Z , we have $f^*D = \text{pr}_{1*}(\text{pr}_2^*D \cdot \Gamma)$.
- (3) The map f^* induces a homomorphism $f^*: N^1(Z) \rightarrow N^1(X)$. This definition of pull-back coincides the definition in [16].

For a dominant rational self-map $f: X \dashrightarrow X$, let $p: Y \rightarrow X$ be a suitable blow-up of X with an ideal sheaf \mathcal{I} whose support is the indeterminacy locus I_f . More precisely, take an embedding $i: X \rightarrow \mathbb{P}^N$. Then the linear system defining the morphism $i \circ f: X \setminus I_f \rightarrow \mathbb{P}^N$ is uniquely extended to a linear system on X . Then we can take \mathcal{I} to be the base ideal of this linear system. Then there exists a surjective morphism $g: Y \rightarrow X$ such that $g = f \circ p$. Using this setting, we can argue as in the proof of Theorem 3.2.

The only non-trivial point is the following. In the proof, we need to bound height functions associated with numerically zero divisors. Precisely, we need the inequality (20). On a smooth projective variety, this is well-known (see for example [6]). Now we need this inequality on Y , which is possibly singular. Actually, this inequality holds on any projective variety.

Lemma B.2 (see for example [11, Theorem 9.5.4]). *Let Y be a normal projective variety over an algebraically closed field. Then there exists a morphism $\alpha: Y \rightarrow A$ with A is an Abelian variety with the following property. For any line bundle L on Y which is algebraically equivalent to zero, there exists a line bundle M on A which is algebraically equivalent to zero such that $L \simeq \alpha^*M$.*

By this lemma and the argument in the proof of [6, Theorem B.5.9], we can easily prove the following.

Proposition B.3. *Let Y be a projective variety over K and E, H divisors on Y with E numerically equivalent to zero and H ample. Fix height functions h_E, h_H associated with these divisors with $h_H \geq 1$. Then there exists a positive constant $C > 0$ such that*

$$|h_E| \leq C\sqrt{h_H}$$

on $Y(K)$.

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